A FEW GRAPHICAL METHODS.

By S. Douglas Killam.

Graphical methods in mathematics are not only of practical value, but are of great interest as they give the student a new viewpoint from which to consider, discuss, or solve some problem in pure mathematics. I will consider here an application of graphical multiplication to the solution of the quadratic equation $ax^2+bx+c=0$; the cubic equation $ax^3+bx^2+cx+d=0$; and in general the polynomial

$$ax^n+bx^{n-1}+\cdots+rx+s=0.*$$

Suppose we have two numbers a and b and we desire to find graphically the product $a\cdot b$. We mark off (Fig. 1) on a straight

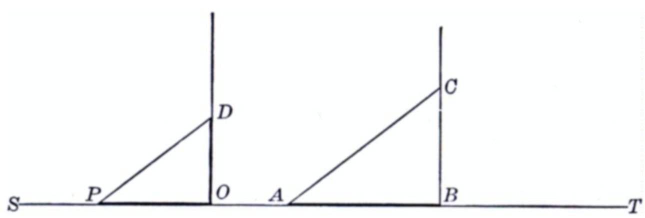

Fig. 1.

line ST, the distance $OP=$ unity, and $AB=b$. At O and B erect perpendiculars to ST and mark off $OD=a$. Through A draw AC parallel to PD, and we have BC equal to the product $a\cdot b$ measured on the same scale as b. We see this from the similar triangles DOP and CBA; where $CB/b=a/1$. If a is a negative number we measure OD downwards, and we have BC going downwards, and therefore represents $a\cdot b$ which is a negative number. If both a and b are negative we measure OD

* This method was first discovered by Captaine Lill, "Résolution graphique des équations numérique d'un degré quelconque à une inconnue," *Nouv. Ann. de math.*, 1867–1868.

downwards and AB to the left of A. This gives us a positive product represented by BC measured upwards.

To divide graphically we need only to reverse our process. We mark off $AB = b$ (Fig. 1) and $BC = a$; then through P we draw PD parallel to AC; and OD represents graphically the number a/b.

This method of graphical multiplication and division gives us a very simple method of representing the laws of multiplication and division.

Now let us consider the function $f(x) = ax^2 + bx + c$; and try to find graphically the value of $f(p)$ where p is any positive or negative real number. As before we mark off $OP =$ unity (Fig. 2) and $AB = a$. We then complete our construction as

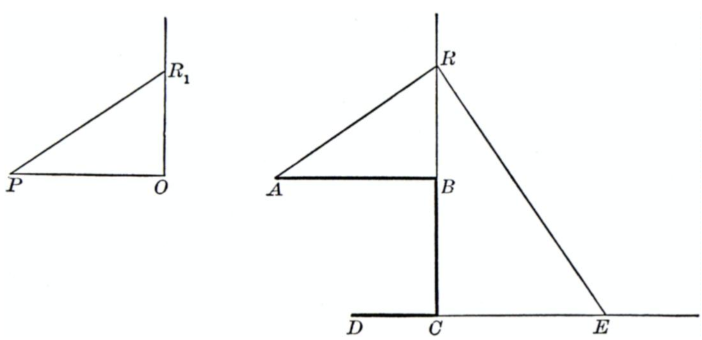

FIG. 2.

in Fig. 1, and see that BR represents in length $a \cdot p$. Produce RB to C making $BC = b$, then $RC = ap + b$. Through C draw CE perpendicular to BC, and through R draw RE perpendicular to AR; then from similar $\triangle$s, EC represents $p(CR) = ap^2 + bp$. Produce EC to D, making $CD = C$; then $ED = ap^2 + bp + c$, which equals $f(p)$ our required result. If a is negative we measure AB to the left of A; if b is negative we measure BC upwards; and if c is negative we measure CD to the right. Now if p is a root of the equation $ax^2 + bx + c = 0$, then $ED = 0$, that is E and D coincide; so that in order to find the value p which makes $ap^2 + bp + c = 0$ we reverse our construction, starting at D or E, which coincide, and going backwards. Notic-

ing that the angle ARD was constructed equal to 90° we have a very simple way of finding the point R. On AD as diameter (Fig. 3) construct a circle, and where this circle cuts BC pro-

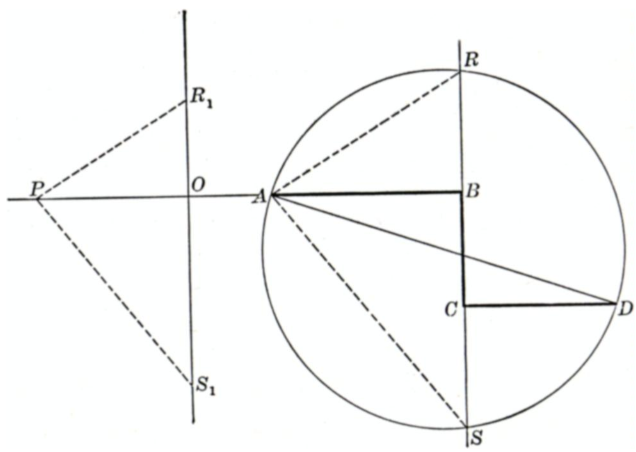

FIG. 3. Graphical Solution of $5x^2+3x-4=0$. Case 1, 2 real roots (unequal).

duced is one of our required points R. Now through P we draw PR_1 parallel to AR, and R_1O represents graphically one of the roots of the quadratic equation $ax^2+bx+c=0$. Our circle cuts BC produced in another point S_1 and if we draw PS_1 paralell to AS, we have OS_1 representing the second root of $ax^2+bx+c=0$.

Proof. Let p_1 and p_2 be the roots represented by R_1O and OS_1 respectively. Then

$RB=ap_1$,	$SB=ap_2$,
$RC=ap_1+b$,	$SC=ap_2+b$,
$DC=ap_1^2+bp_1$,	$DC=ap_2^2+bp_2$,
$DC+c=ap_1^2+bp_1+c=0$,	$DC+c=ap_2^2+bp_2+c=0$.

p_1 is a positive root being measured upwards; and p_2 is a negative root being measured downwards.

Corresponding to the three cases of the roots of a quadratic equation, (1) unequal real roots, (2) equal real roots, and (3) imaginary roots, we have three graphical constructions which illustrate clearly the three different cases; and this graphical interpretation gives the student a new idea of imaginary roots.

Case (1) (Fig. 3). The circle on AD as diameter cuts BC produced in *two* distinct points which give us our two unequal real roots.

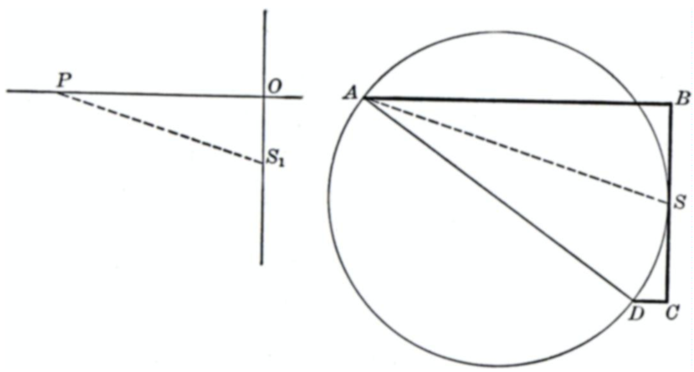

FIG. 4. Graphical Solution of $9x^2 + 6x + 1 = 0$. Case 2, equal roots.

Case (2) (Fig. 4). The circle on AD as diameter just touches BC in one point (or two coinciding points) which gives us our two equal roots.

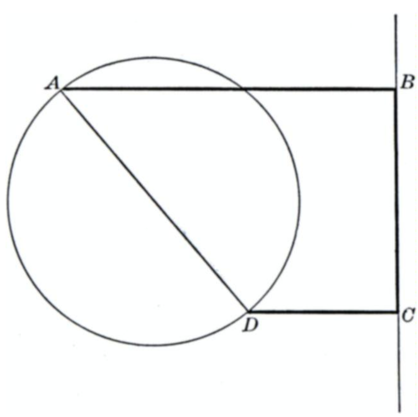

FIG. 5. Graphical Solution of $9x^2 + 6x + 4 = 0$. Case 3, imaginary roots.

Case (3) (Fig. 5). The circle on AD as diameter does not touch or cut BC, and this corresponds to the case where the roots are imaginary. The accuracy of this method depends on the accuracy of our drawing and on the size of our unit of measurement. With a little practice and a unit large enough to make our drawing distinct, the results obtained will be accurate enough for most problems which involve the finding of the roots of a quadratic equation.

The roots (if real) of a cubic equation can be found in the same way. Fig. 6 illustrates the method of finding the value of

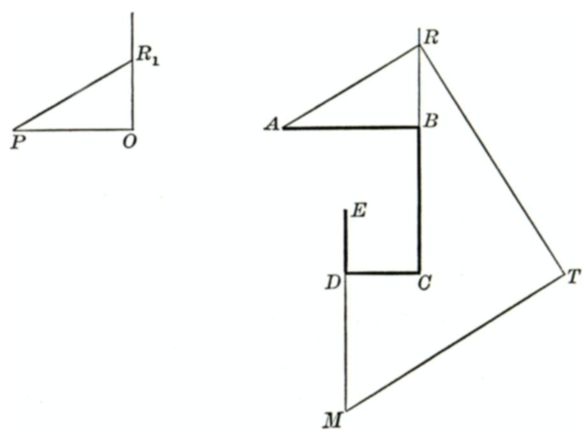

FIG. 6.

$ap^3 + bp^2 + cp + d$. If p is a root of our cubic equation M and E coincide, and we simply need to reverse our process.

$$
\begin{aligned}
BR &= ap. \\
RC &= ap + b. \\
TC &= ap^2 + bp. \\
TD &= ap^2 + bp + c. \\
MD &= ap^3 + bp^2 + cp. \\
ME &= ap^3 + bp^2 + cp + d.
\end{aligned}
$$

Starting with the point E we have no simple method as with the quadratic equation of finding the roots. We must take some value p which we think approximates a root, and complete our

construction, and see if M and E coincide. If not take a better approximation and so on until M and E coincide.

In general the graphical solution of the equation

$$ax^n + bx^{n-1} + \cdots + rx + s = 0$$

is the same as with the cubic equation, but the greater the value of n the more complicated our work becomes.

Two linear equations with two unknowns.—

$$a_1x + b_1y + c_1 = 0 \qquad (1)$$
$$a_2x + b_2y + c_2 = 0 \qquad (2)$$

A very interesting graphical solution of the above problem was first given in an article by "van den Berg." His method is as follows.

On a line OD perpendicular to a line MN (Fig. 7) we mark

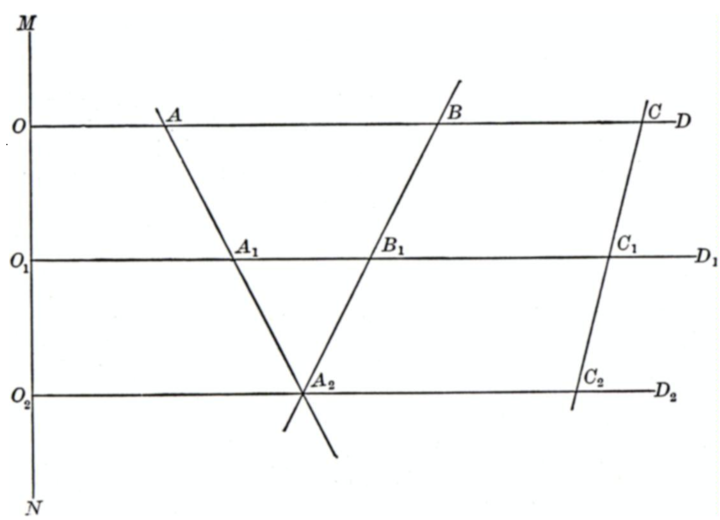

FIG. 7. Graphical Solution of $2x + 4y + 3 = 0$, $3x + 2y + 3.5 = 0$.

off $OA = a_1$; $AB = b_1$; and $BC = c_1$; on O_1D_1 also perpendicular to MN we mark off $O_1A_1 = a_2$; $A_1B_1 = b_2$; and $B_1C_1 = c_2$. Join AA_1; BB_1; and CC_1. Through A_2 the point where AA_1 and BB_1 meet draw O_2D_2 perpendicular to MN; then A_2C_2/O_2A_2

represents graphically the value of x which satisfies the equations above.

Proof.—Multiply (2) by λ and add to (1). Then $(a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2) = 0$. We choose λ so that $b_1 + \lambda b_2 = 0$, that is $\lambda + - b_1/b_2$. From similar $\triangle$s we see that $x = A_2C_2/O_2A_2$.

We could have eliminated the term in x by drawing a line through the point where A_1A_2 and MN meet. If AA_1 and BB_1 are parallel we must do this. If MN; A_1A; and B_1B are all parallel no solution exists. If MN; A_1A; B_1B and C_1C are all parallel then an infinite number of solutions exist. These graphically different cases can easily be interpreted in algebraic form. For three linear equations with three unknowns, we first eliminate one variable, and have left our problem with two variables. In the same way we solve n linear equations with n unknowns.

THE UNIVERSITY OF ROCHESTER,
ROCHESTER, N. Y.